NORTH-HOLLAND

Variational Analysis of an Extended Eigenvalue Problem*

Ronald J. Stern

Department of Mathematics and Statistics
Concordia University
Montreal, Quebec H4B 1R6, Canada

and

J. J. Ye

Centre de recherches mathématiques
Université de Montréal
Montréal, Québec H3C 3J7, Canada

Dedicated to Professor Adi Ben-Israel on the occasion of his sixtieth birthday.

Submitted by Richard A. Brualdi

ABSTRACT

For a symmetric matrix $B \in R^{n\times n}$ and a vector $a \in R^n$, the maximal extended eigenvalue

$$\lambda(a) := \max\{\lambda : \exists x \in R^n \text{ s.t. } (B - \lambda I)x = a,\ x^t x = 1\},$$

is known to arise in optimality conditions for the mathematical programming problem $P(a)$ given by

$$\max\{x^t Bx - 2a^t x : x^t x = 1\},$$

as well as in extended Rayleigh-Ritz type results pertaining to the one parameter

*Research supported by Natural Sciences Engineering Research Council Canada operating grant A4641.

LINEAR ALGEBRA AND ITS APPLICATIONS 220:391–417 (1995)

0024-3795/95/$9.50
SSDI 0024-3795(92)00016-F

family of nonsymmetric border perturbations of B given by

$$A(t;a) := \begin{pmatrix} B & a \\ -a^t & t \end{pmatrix}.$$

Nonsmooth analysis is employed in order to describe the function $\lambda(\cdot)$, with special emphasis on its sensitivity near the origin. Further connections with $P(a)$ are drawn.

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1. INTRODUCTION

Let B be a real symmetric $n \times n$ matrix. For each vector $a \in R^n$, consider the mathematical programming problem $P(a)$ given by

$$\begin{array}{lrcl} \text{maximize} & \mu_a(x) & := & x^t Bx - 2a^t x \\ \text{subject to} & x^t x & = & 1. \end{array}$$

The first order Lagrange necessary condition for a feasible point $\hat{x}$ to give the maximum in $P(a)$ is the existence of $\hat{\lambda} \in R$ such that

$$(B - \hat{\lambda} I)\hat{x} = a. \tag{1.1}$$

It is in fact known that necessarily $\hat{\lambda} = \lambda(a)$, where

$$\lambda(a) := \max\{\lambda : \exists x \in R^n \text{ s.t. } (B - \lambda I)x = a,\ x^t x = 1\} \tag{1.2}$$

denotes the maximal Lagrange multiplier of the problem, or equivalently, the *maximal extended eigenvalue* of the pair $\{B, a\}$. Note that we regard the maximum in (1.2) as being taken over the feasible set

$$F(a) := \{(\lambda, x) \in R^{n+1} : (B - \lambda I)x = a,\ x^t x = 1\}, \tag{1.3}$$

which is nonempty, by the existence of Lagrange multipliers. The maximum in (1.2) is therefore attained, because $F(a)$ is compact (straightforward), and the function $g : R^{n+1} \to R$ given by $g(\lambda, x) := \lambda$ is obviously continuous.

In the present work, we utilize the techniques of nonsmooth analysis, as presented in Clarke [3, 2], in order to describe the function $\lambda(\cdot)$. We will note connections with the problem $P(a)$, and in particular with its associated *value function* $V(\cdot)$, given by

$$V(a) := \max\{\mu_a(x) : x^t x = 1\}. \tag{1.4}$$

Of particular interest to us is the behavior of $\lambda(\cdot)$ near the origin. At the outset, it is readily noted that $\lambda(0)$ is the maximal eigenvalue of B, which, in view of the classical Rayleigh-Ritz theorem, also equals $V(0)$. Our results constitute part of sensitivity analysis, since in (1.2) one may consider the vector a as a perturbation of 0; in this regard, the generalized gradient $\partial\lambda(0)$ can be used in order to provide a measure of how much the maximal extended eigenvalue $\lambda(a)$ differs from the maximal eigenvalue b_1 of B.

The problem $P(a)$ has been extensively discussed in the literature. The theory has been developed by Forsythe and Golub [5], Golub [8], Gander [6], Sorensen [11], and Gander, Golub, and Von Matt [7]. Problems related to $P(a)$ occur in iterative steps of a class of numerical procedures for unconstrained optimization called "trust region" methods. Also, note that the problem of minimizing $||Ax - c||$ subject to $x^t x = 1$ can be cast into the form of $P(a)$; here $||\cdot||$ denotes the euclidean norm. Efficient algorithms for numerically solving such problems have been given by Moré and Sorensen [10] and by Golub and Von Matt [9]. Furthermore, in Stern and Wolkowicz [12], connections were established between stationarity properties of the problem $P(a)$ and the spectral structure of the one-parameter family of nonsymmetric border perturbations of B given by

$$A(t; a) := \begin{pmatrix} B & a \\ -a^t & t \end{pmatrix}.$$

In the next section, we provide the basic material from nonsmooth analysis that will be utilized. Then in Section 3 we review relevant aspects of the theory of the mathematical programming problem $P(a)$, and give a preliminary description of the function $\lambda(\cdot)$. The main results are presented in Section 4. There proximal analysis is employed in order to derive a multiplier result on the sensitivity of $\lambda(\cdot)$ near 0. This leads to the following sequence of results:

(i) We establish continuous differentiabilty on a "large" set of points, and derive a useful formula for $\bigtriangledown\lambda(\cdot)$ on that set.
(ii) It is proven that $\lambda(\cdot)$ is Lipschitz near 0, and the generalized gradient $\partial\lambda(0)$ is characterized.

(iii) We prove that $\lambda(\cdot)$ is regular at 0, and directional derivatives of $\lambda(\cdot)$ at 0 are characterized.

In addressing items (ii) and (iii) above, we first settle things for the special case where b_1 is a simple eigenvalue, and then pass to the general situation via a perturbation method. Concluding comments include a monotonicity property of $\lambda(\cdot)$.

2. NONSMOOTH ANALYSIS BACKGROUND

In this section we shall give a concise review of the material on nonsmooth analysis which will be required. Our references are Clarke [3, 2].

Let C be a nonempty closed set in R^n, and let $c \in C$. Then $w \in R^n$ is called a *perpendicular* to C at c provided that

$$d_C(c+w) = ||w|| > 0,$$

where $d_C(\cdot)$ denotes the distance to C. The set of *proximal normals* to C at c is the set

$$\Pi_C(c) := \{\alpha w : w \text{ is a perpendicular to } C \text{ at } c, \ \alpha \geq 0\}.$$

It is readily noted that $\Pi_C(c) = \{0\}$ if c is an interior point of C, and that the set of boundary points of C admitting nonzero proximal normals is dense in the boundary of C. This leads to the following definition: The *normal cone* to C at c is the set

$$N_C(c) := \overline{\text{co}}\{\lim \xi_i : \xi_i \in \Pi_C(c_i), \ c_i \to c\}.$$

Here co means convex hull, and the overbar denotes closure.

Let $f : R^n \to R$ be a continuous function. The *generalized gradient* of $f(\cdot)$ at $x \in R^n$ is the (possibly empty) closed convex set

$$\partial f(x) := \{\zeta : (\zeta, -1) \in N_{epi\, f}(x, f(x))\},$$

where the (closed) set

$$\text{epi}\, f := \{(x, r) \in R^n \times R : r \geq f(x)\}$$

is the *epigraph* of $f(\cdot)$. We shall say that $f(\cdot)$ is *Lipschitz of rank K near* x provided that there exists an open neighborhood Γ of x such that

$$|f(x^1) - f(x^2)| \leq K||x^1 - x^2|| \ \ \forall x^1, x^2 \in \Gamma.$$

When the Lipschitz constant K is not directly relevant, we will just say that $f(\cdot)$ is Lipschitz near x. Also, recall that a set valued mapping $h : R^n \to R^n$ is said to be *upper-semicontinuous* at x if the following holds: Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$||x - y|| < \delta \Longrightarrow h(y) \subset h(x) + B_\varepsilon(0),$$

where $B_\varepsilon(0)$ denotes the open ball of radius ε centered at 0.

The following proposition summarizes requisite facts regarding the generalized gradient.

PROPOSITION 2.1.

1. *A sufficient condition for $\partial f(x)$ to be nonempty is that x is a local minimum of $f(\cdot)$.*
2. *$\partial f(x)$ is nonempty and bounded if and only if $f(\cdot)$ is Lipschitz near x. In this case, the following hold:*
   (a) *The generalized gradient satisfies*

   $$\partial f(x) = \{\zeta \in R^n : f^o(x; v) \geq \langle v, \zeta \rangle \ \ \forall v \in R^n\},$$

   where

   $$f^o(x; v) := \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}$$

   denotes the generalized directional derivative *of $f(\cdot)$ at x in the direction v.*

   (b) *One has*

   $$f^o(x; v) = \max\{\langle \zeta, v \rangle : \zeta \in \partial f(x)\}.$$

   (c) *The function $f(\cdot)$ is continuously differentiable on an open neighborhood of x if and only if $\partial f(x)$ is single valued on that neighborhood.*

   (d) *Furthermore,*

   $$\partial f(x) = \text{co}\{\lim \nabla f(x_i) : x_i \to x, \ x_i \notin S, \ x_i \notin \Omega_f\},$$

   where S is any set of Lebesgue measure 0, and Ω_f denotes the set where $f(\cdot)$ fails to be differentiable. The set Ω_f has measure 0 as well.

   (e) *For any $\alpha \in R$, one has*

   $$\partial(\alpha f)(x) = \alpha \partial f(x).$$

(f) $\partial f(\cdot)$ is upper-semicontinuous at x.

A vector $\zeta \in R^n$ is called a *proximal subgradient* of $f(\cdot)$ at x provided that there exists $\sigma > 0$ such that for all y in some open neighborhood of x we have

$$f(y) - f(x) + \sigma ||y - x||^2 \geq \langle \zeta, y - x \rangle.$$

The set of all proximal subgradients of $f(\cdot)$ at x is denoted $\partial^\pi f(x)$. It can be shown that the set of points admitting proximal subgradients is dense in R^n. The *presubgradient* of f at x is defined to be the (possibly empty) set

$$\hat{\partial} f(x) := \{\lim \zeta_i : \zeta_i \in \partial^\pi f(x_i),\ x_i \to x\}.$$

The *singular presubgradient* of $f(\cdot)$ at x is the set

$$\hat{\partial}^\infty f(x) := \{\lim t_i \zeta_i : \zeta_i \in \partial^\pi f(x_i),\ x_i \to x,\ t_i \downarrow 0\}.$$

The latter set always contains $\{0\}$.

The generalized gradient, presubgradient, and singular presubgradient are related by the following useful proposition.

PROPOSITION 2.2. *The function f is Lipschitz near x if and only if $\hat{\partial}^\infty f(x) = \{0\}$. In that case we have $\partial f(x) = \text{co}\{\hat{\partial} f(x)\}$. In general we have*

$$\partial f(x) = \overline{\text{co}}\{\hat{\partial} f(x) + \hat{\partial}^\infty f(x)\}, \tag{2.5}$$

(with $\partial f(x) = \phi$ if and only if $\hat{\partial} f(x) = \phi$.)

3. BACKGROUND FOR $P(A)$ AND PRELIMINARY RESULTS

In [11], Sorensen proved that a necessary and sufficient condition for a feasible vector $\hat{x}$ to give the maximum in the problem $P(a)$ is that there exists $\hat{\lambda} \in R$ such that (1.1) holds, and such that

$$B - \hat{\lambda} I \leq 0, \tag{3.6}$$

where $\leq$ denotes negative semidefiniteness. Furthermore, if

$$B - \hat{\lambda} I < 0, \tag{3.7}$$

then the optimal point $\hat{x}$ is unique, with $<$ denoting negative definiteness. (See also Section 5.2 in Fletcher [4].)

Let the spectrum of B be given by

$$b_1 \geq b_2 \geq \cdots \geq b_n.$$

Then (3.6) becomes

$$b_1 - \lambda(a) \leq 0. \tag{3.8}$$

With the maximal Lagrange multiplier $\lambda(a)$ defined as in (1.2), it now follows that the set of optimal solutions $X(a)$ of the problem $P(a)$ is the compact set

$$X(a) := \{x \in R^n : [B - \lambda(a)I]x = a,\ x^t x = 1\}. \tag{3.9}$$

Suppose that

$$a \notin R(B - b_1 I), \tag{3.10}$$

where $R(\cdot)$ denotes range. Then we must have

$$b_1 - \lambda(a) < 0, \tag{3.11}$$

and therefore $X(a)$ is the singleton $x(a) = [B - \lambda(a)I]^{-1}a$. In view of the feasibility condition $x(a)^t x(a) = 1$, it follows that $\lambda(a)$ is the maximal solution λ of the equation

$$1 - a^t(B - \lambda I)^{-2}a = 0. \tag{3.12}$$

The discussion will now be extended to the situation where possibly $a \in R(B - b_1 I)$. Following Gander, Golub, and Von Matt [7], we initially shall make the assumption that B is diagonal; that is,

$$B = \operatorname{diag}\{b_1, b_2, \ldots, b_n\}.$$

We introduce the index sets

$$J(a) = \{j : a_j \neq 0,\ 1 \leq j \leq n\}$$

and

$$J(b) = \{j : b_j = b_1\} = \{b_1, b_2, \ldots, b_k\},$$

where k is the algebraic multiplicity of b_1. The so-called secular function for $P(a)$ is defined as

$$s_a(\lambda) := 1 - \sum_{j \in J(a)} \frac{a_j^2}{(b_j - \lambda)^2}. \tag{3.13}$$

In describing $X(a)$, two main cases are to be considered.

Case 1: Suppose there exists $\hat{\lambda} > b_1$ such that $s_a(\hat{\lambda}) = 0$. The preceding discussion shows that this surely is the case if $a \notin R(B - b_1 I)$ that is, if $J(a) \cap J(b) \neq \phi$. However, case 1 can also occur with $a \in R(B - b_1 I)$. In either contingency, the derivative $s'_a(\lambda) > 0$ for all $\lambda > b_1$, which implies that $\hat{\lambda} = \lambda(a)$, and the unique solution to the problem $P(a)$ is $x(a) = [B - \lambda(a)I]^{-1}a$.

Case 2: Now suppose there does not exist $\lambda > b_1$ such that $s_a(\lambda) = 0$. Then $a \in R(B - b_1 I)$ [equivalently, $J(a) \cap J(b) = \phi$], from which it follows that $\lambda(a) = b_1$. Also, it is clear that $s_a(b_1) \geq 0$. Let

$$w(a) := \sum_{j \in J(a)} \frac{a_j^2}{(b_j - b_1)^2} = 1 - s_a(b_1).$$

Then $0 \leq w(a) \leq 1$, and we consider two subcases.

Case 2′: If $w(a) < 1$, then the set of optimal solutions $X(a)$ consists of vectors x such that

$$x_j = \frac{a_j}{b_j - b_1} \qquad \forall j \in J(a)$$

and

$$\sum_{j \in J(b)} x_j^2 = 1 - w(a).$$

Case 2″: If $w(a) = 1$, then $X(a)$ is the singleton $x = x(a)$ such that

$$x_j = \frac{a_j}{b_j - b_1} \qquad \forall j \in J(a)$$

and

$$x_j = 0 \qquad \forall j \in J(b).$$

Terminology: If a is such that case 1 holds, then we say $a \in C(1)$. Similarly, if case $2'$ or $2''$ holds, then we say $a \in C(2')$ or $a \in C(2'')$, respectively.

REMARK 3.1.

1. By studying the behavior of the secular function, one can show that the set of all Lagrange multipliers for the problem $P(a)$,

$$M(a) := \{\lambda : (B - \lambda I)x = a,\ x^t x = 1\}$$

is a discrete set, containing at least 2 and at most $2n$ elements. (This follows from the analysis in Forsythe and Golub [5] and Gander, Golub, and Von Matt [7].)

2. In Stern and Wolkowicz [12] it was shown that a sufficient condition for the spectrum of the matrix $A(t;a)$ to be real is $t \geq V(a)$. Furthermore, $\lambda(a)$ is the maximal eigenvalue of

$$A(V(a);a) = \begin{pmatrix} B & a \\ -a^t & V(a) \end{pmatrix},$$

and what is more, for any $x \in X(a)$, the vector $\binom{x}{1}$ is an eigenvector of $A(V(a);a)$ belonging to $\lambda(a)$. The analysis in [12] shows that the algebraic multiplicity of $\lambda(a)$ is $k+2$, while the geometric multiplicity is $k+1$, where k is the algebraic multplicity of $\lambda(a)$ as an eigenvalue of B.

3. Since the equation $s_a(\lambda(a)) = 0$ holds for $a \in C(1)$, we have

$$\lambda(a) = b_1 + ||a|| \qquad \forall a \in N(B - b_1 I), \tag{3.14}$$

where $||\cdot||$ denotes the euclidean norm. If the algebraic multiplicity of b_1 is n, then $N(B - b_1 I) = R^n$, and the explicit formula (3.14) holds everywhere.

REMARK 3.2. For $\gamma \geq b_1$, we introduce the level set

$$L(\gamma) := \{a \in R^n : \lambda(a) = \gamma\}.$$

1. For $\gamma > b_1$, we have

$$L(\gamma) = \{a \in R^n : a^t(B - \gamma I)^{-2} a = 1\},$$

which is the boundary of an ellipsoid with nonempty interior. Furthermore, $L(\gamma) \subset C(1)$.

2. In view of the preceding discussion, we also see that

$$L(b_1) = \{a \in R(B - b_1 I) : \sum_{j \notin J(b)} \frac{a_j^2}{(b_j - b_1)^2} \leq 1\},$$

which is an ellipsoid contained in $R(B - b_1 I)$, whose relative interior and relative boundary are $C(2')$ and $C(2'')$, respectively.

3. If the algebraic muliplicity of b_1 is n, then

$$R(B - b_1 I) = C(2) = L(b_1) = \{0\},$$

and for each $\gamma > b_1$, the level set is the circle

$$L(\gamma) = \{a \in R^n : \gamma = b_1 + ||a||\}.$$

Figures 1 and 2 for $B = \text{diag}\{1, 0\}$ and $B = \text{diag}\{1, 1\}$ serve to motivate the analysis in subsequent sections.

REMARK 3.3. With regard to the two figures, it is clear that $\lambda(\cdot)$ is not concave for either B. For $B = \text{diag}\{1, 1\}$, we see that $\lambda(\cdot)$ is convex, since (3.14) holds everywhere. However, for $B = \text{diag}\{1, 0\}$, the function is *not* convex. This is seen as follows. Let

$$S = \{(a_1, 1) : a_1 \geq 0\}$$

and

$$a(\gamma) = S \cap L(\gamma).$$

The midpoint of the line segment going from the point $a(1) = (0, 1)$ to $a(3)$ lies on a level set $L(\gamma)$ with $\gamma > 2$. This shows that convexity is precluded.

Now let us drop the assumption that B is diagonal. Let $U \in R^{n \times n}$ be a unitary matrix such that

$$U^t B U = \hat{B} = \text{diag}\{b_1, b_2, \ldots, b_n\},$$

and consider the mathematical programming problem $\hat{P}(\hat{a})$ given by

$$\begin{array}{lll} \text{maximize} & \hat{\mu}_{\hat{a}}(\hat{x}) & := \hat{x}^t \hat{B} \hat{x} - 2\hat{a}^t \hat{x} \\ \text{subject to} & \hat{x}^t \hat{x} & = 1, \end{array}$$

where $\hat{a} = U^t a$. Then $\hat{x} = U^t x$ is feasible for $\hat{P}(\hat{a})$, and what is more, we have

$$\mu_a(x) = \hat{\mu}_{\hat{a}}(U^t x). \tag{3.15}$$

We shall now see precisely how the various sets associated with $P(a)$ are expressible as transformations of the corresponding sets in the "diagonalized" problem $\hat{P}(\hat{a})$. Let us define

$$\hat{\lambda}(\hat{a}) := \max\{\hat{\lambda} \in R : \exists \hat{x} \in R^n \text{ s.t. } (B - \hat{\lambda} I)\hat{x} = \hat{a},\ \hat{x}^t \hat{x} = 1\}. \tag{3.16}$$

Then the set of optimal solutions to $\hat{P}(\hat{a})$ is given by

$$\hat{X}(\hat{a}) := \{\hat{x} \in R^n : [\hat{B} - \hat{\lambda}(\hat{a}) I]\hat{x} = \hat{a},\ \hat{x}^t \hat{x} = 1\}. \tag{3.17}$$

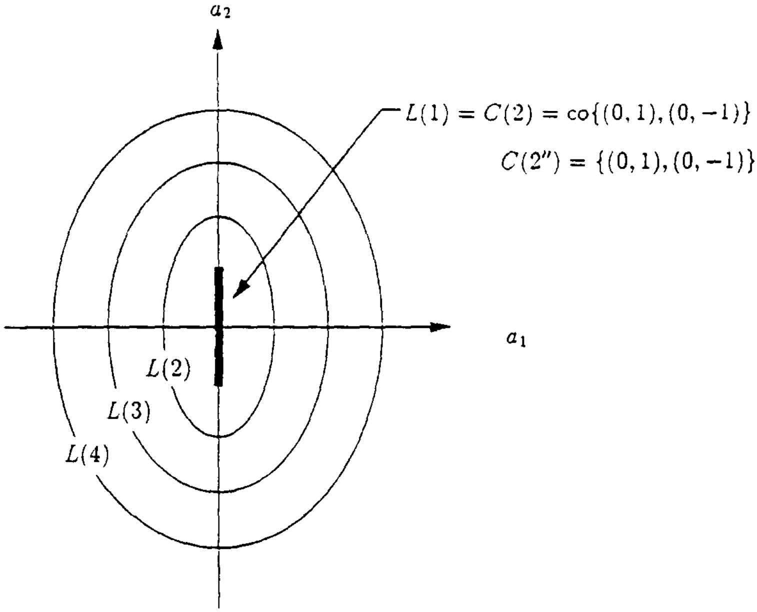

FIG. 1. Level sets for $B = \mathrm{diag}\{1,0\}$

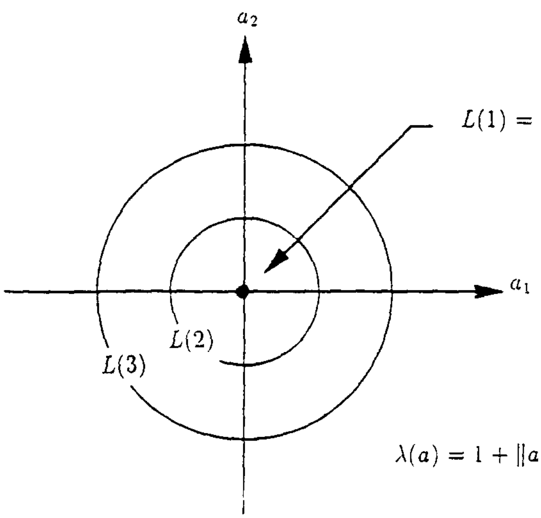

FIG. 2. Level sets for $B = \mathrm{diag}\{1,1\}$

For each $\gamma \geq b_1$ we define the level set

$$\hat{L}(\gamma) := \{\hat{a} \in R^n : \lambda(\hat{a}) = \gamma\}.$$

We ask the reader to check that

$$X(a) = U\hat{X}(U^t a), \tag{3.18}$$

$$\lambda(a) = \hat{\lambda}(U^t a), \tag{3.19}$$

and that for each $\gamma \geq b_1$,

$$L(\gamma) = U\hat{L}(\gamma). \tag{3.20}$$

More terminology: For the problem $P(a)$, we shall henceforth denote by $C(1)$, $C(2')$, and $C(2'')$ the transformations, under U, of the corresponding sets in the transformed problem $\hat{P}(\hat{a})$.

Prior to presenting our main results, we pause in order to gather some easily derived properties of the functions $\lambda(\cdot)$, $X(\cdot)$ and the value function $V(\cdot)$ defined in (1.4).

PROPOSITION 3.1.

1. *For each $a \in R^n$, one has*

$$V(a) = \lambda(a) - a^t x \qquad \forall x \in X(a). \tag{3.21}$$

2. *Furthermore,*

$$V(0) = \lambda(0) = b_1 \tag{3.22}$$

and

$$V(a) > \lambda(a) \geq b_1 \qquad \forall a \neq 0. \tag{3.23}$$

3. *The function $V(\cdot)$ is convex and finite valued on R^n, and*

$$\partial V(a) = -2\,\mathrm{co}\{X(a)\} \qquad \forall a \in R^n. \tag{3.24}$$

4. *The compact set valued functions $X(\cdot)$ and $\partial V(\cdot)$ are upper-semicontinuous on R^n. These functions are single valued on $C(1) \cup C(2'')$ and multivalued on $C(2')$.*
5. *The function $\lambda(\cdot)$ is continuous on R^n.*
6. *The function $V(\cdot)$ is continuously differentiable at a if and only if $a \in C(1) \cup C(2'')$.*

Proof: We begin by noting that in proving each of the statements, one may without loss of generality assume that B is diagonal. Equation (3.21) in part (1) results upon combining the facts that for each $x \in X(a)$ we have

$$[B - \lambda(a)I]x = a$$

and

$$V(a) = x^t Bx - 2a^t x.$$

In part (2), we obtain (3.22) directly from (3.21). The strict inequality in (3.23) is a direct consequence of the case-by-case analysis we did for diagonal B, while $\lambda(a) \geq b_1$ is just a restatement of (3.8). In part (3), $V(a)$ is clearly finite for each $a \in R^n$, since it is the maximum of a continuous function over a compact set. The asserted convexity follows from the fact that $V(\cdot)$ is the pointwise maximum of a family of affine functions. Equation (3.24) follows from Corollary 1 of Theorem 2.8.2 in Clarke [2]. As for part (4), the asserted upper semicontinuity of $X(\cdot)$ follows from a well-known result on marginal maps; see e.g. Theorem 6 on p. 53 in Aubin and Cellina [1]. Upper semicontinuity for $\partial V(\cdot)$ then follows from part (3). The rest of part (4) is a consequence of our discussion of cases 1 and 2 above.

We now turn to part (5). From (3.21), we see that the set of reals

$$G(a) := \bigcup \{a^t x : x \in X(a)\}$$

is a singleton for each $a \in R^n$. Since $X(\cdot)$ is upper-semicontinuous, it follows that $G(\cdot)$ is continuous. Now, $V(\cdot)$ is continuous, since it is finite valued and convex on R^n. Therefore (3.21) implies that $\lambda(\cdot)$ is continuous. Finally, part (6) follows from Proposition 2.1 (2)(c). ∎

REMARK 3.4. With regard to the preceding proposition, we have the following:

1. Equation (3.22) is the classical Rayleigh-Ritz theorem.
2. The upper semicontinuity of $X(\cdot)$ may alternatively be deduced directly from the foregoing discussion of cases 1 and 2.
3. The upper semicontinuity of $\partial V(\cdot)$ also follows from Proposition 2.1 (2)(f).

By the light of Proposition 3.1, we see that the function $\lambda(\cdot)$ is everywhere bounded below by b_1 and above by the function $V(\cdot)$, which is locally Lipschitz, since it is convex. Furthermore, $\lambda(0) = V(0) = b_1$. While these facts alone are insufficient to establish Lipschitz continuity of $\lambda(\cdot)$ near 0, a sketch of the situation leads one to realize that it would take extremely bad behavior of $\lambda(\cdot)$ for the local Lipschitz property to fail. In the next section we will see that such behavior in fact does not occur; see Theorem 4.2 below.

4. MAIN RESULTS

For any vector $a \in R^n$, consider the mathematical programming problem $\tilde{P}(a)$ given by

$$\begin{array}{lrcl} \text{minimize} & f(\lambda, x) & := & -\lambda \\ \text{subject to} & (B - \lambda I)x & = & a, \\ & x^t x & = & 1. \end{array}$$

The optimal value for the problem $\tilde{P}(a)$ is $-\lambda(a)$, and the (compact) set of optimal solutions is

$$\Sigma^a := \{(\lambda(a), x) : x \in X(a)\}.$$

Let (λ, x) be feasible for $\tilde{P}(a)$, and let $l \geq 0$. Motivated by the variational analysis of value functions in [3] and [2], for $l \in R$ we introduce the multiplier set

$$M^l(\lambda, x) = \{r \in R^n : \exists k \in R \text{ s.t. } (B - \lambda I)r + kx = 0,\ r^t x = l\}.$$

The following sensitivity theorem for the problem $\tilde{P}(a)$ will prove to be very useful in the subsequent analysis.

THEOREM 4.1. *For every $a \in R^n$ one has*

$$\hat{\partial}^\infty(-\lambda)(a) \subset M^0(\Sigma^a) \tag{4.25}$$

and

$$\partial(-\lambda)(a) \subset \text{co}\{M^1(\Sigma^a)\}. \tag{4.26}$$

Proof: Suppose that $(\lambda(a), x) \in \Sigma^a$, and let $\beta \in \partial^\pi(-\lambda)(a)$. Then there exists $\sigma > 0$ such that for any $\bar{a}$ near a we have

$$\begin{aligned} -\lambda(\bar{a}) - \langle \beta, \bar{a} \rangle + \sigma \|\bar{a} - a\|^2 &\geq -\lambda(a) - \langle \beta, a \rangle \\ &= -\lambda(a) - \langle \beta, [B - \lambda(a)I]x \rangle. \end{aligned}$$

Since $\lambda(\bar{a}) \geq \bar{\lambda}$ for all $(\bar{\lambda}, \bar{x})$ such that $(B - \bar{\lambda}I)\bar{x} = \bar{a}$, $\bar{x}^t\bar{x} = 1$, it follows that $(-\lambda(a), x)$ is a local solution of the following problem:

$$\begin{array}{ll} \text{minimize} & -\bar{\lambda} - \langle \beta, (B - \bar{\lambda}I)\bar{x} \rangle \\ \text{subject to} & \bar{x}^t\bar{x} = 1. \end{array}$$

By the classical Lagrange multiplier rule, we conclude that there exists $\hat{k} \in R$ such that

$$\begin{aligned} -1 + \langle \beta, x \rangle &= 0, \\ [B - \lambda(a)I]\,\beta + \hat{k}x &= 0, \\ x^t x &= 1. \end{aligned}$$

Now let $\hat{\beta} \in \hat{\partial}(-\lambda)(a)$. Then $\hat{\beta} = \lim \beta_i$, where $\beta_i \in \partial^\pi(-\lambda)(a_i)$, and $a_i \to a$. By the preceding arguments, for each β_i and $x_i \in X(a_i)$, there exists $k_i \in R$ such that

$$\begin{aligned} -1 + \langle \beta_i, x_i \rangle &= 0, \\ [B - \lambda(a_i)I]\,\beta + k_i x_i &= 0, \\ x_i^t x_i &= 1. \end{aligned}$$

It is obvious that the sequence $\{k_i\}$ is bounded. We may therefore assume that $k_i \to k$ and $x_i \to \hat{x} \in X(a)$, where we have used the fact that the compact set valued function $X(\cdot)$ is upper-semicontinuous. Then the continuity of $\lambda(\cdot)$ yields

$$\begin{aligned} -1 + \langle \hat{\beta}, \hat{x} \rangle &= 0, \\ [B - \lambda(a)I]\,\hat{\beta} + k\hat{x} &= 0, \\ \hat{x}^t\hat{x} &= 1. \end{aligned}$$

Therefore, $\hat{\beta} \in M^1(\lambda(a), \hat{x})$, and we have shown that

$$\hat{\partial}(-\lambda)(a) \subset M^1(\Sigma^a). \tag{4.27}$$

Now let $\beta^\infty \in \hat{\partial}^\infty(-\lambda)(a)$. Then $\beta^\infty = \lim t_i\beta_i$ as $t_i \downarrow 0$, where $\beta_i \in \partial^\pi(-\lambda)(a_i)$ and $a_i \to a$. We know that for each β_i and each $x_i \in X(a_i)$, there exists $k_i \in R$ such that

$$\begin{aligned} -1 + \langle \beta_i, x_i \rangle &= 0, \\ [B - \lambda(a_i)I]\,\beta_i + k_i x_i &= 0, \\ x_i^t x_i &= 1. \end{aligned}$$

Multiply both sides of the first two equations by t_i, and take the limit as $t_i \downarrow 0$. Arguing similarly to above, we conclude that there exists $\tilde{k} \in R$ and $\tilde{x} \in X(a)$ such that

$$\begin{aligned}\langle \beta^\infty, \tilde{x} \rangle &= 0,\\ [B - \lambda(a)I]\,\beta^\infty + \tilde{k}\tilde{x} &= 0,\\ \tilde{x}^t \tilde{x} &= 1.\end{aligned}$$

Hence $\beta^\infty \in M^0(\Sigma^a)$, which proves (4.25). The containment (4.26) now follows from (2.5), (4.25), (4.27), and the readily verifiable fact that

$$M^1(\Sigma^a) + M^0(\Sigma^a) = M^1(\Sigma^a).$$

■

We will now extract corollaries of Theorem 4.1. The first of these gives a sufficient condition for continuous differentiability of $\lambda(\cdot)$, and formulas for the gradient and directional derivative. We recall that the ordinary *directional derivative* of $f(\cdot)$ at x in the direction v, should it exist, is defined by

$$f'(x;v) := \lim_{t \downarrow 0} \frac{f(x+tv) - f(x)}{t}.$$

COROLLARY 4.1. *The function $\lambda(\cdot)$ is continuously differentiable on the set $C(1)$. For $a \in C(1)$ we have*

$$\nabla \lambda(a) = -\frac{1}{a^t\,[B - \lambda(a)I]^{-3}\,a}\,[B - \lambda(a)I]^{-2}\,a. \tag{4.28}$$

Furthermore, the directional derivative of $\lambda(\cdot)$ at $a \in C(1)$ in the direction v is given by

$$\lambda'(a;v) = -\frac{a^t[B - \lambda(a)I]^{-2}v}{a^t\,[B - \lambda(a)I]^{-3}\,a}. \tag{4.29}$$

Proof: When case 1 holds, $[B - \lambda(a)I]^{-1}$ is invertible and negative definite. Then

$$M^0(\Sigma^a) = \{0\},$$

since the only r simultaneously satisfying $[B - \lambda(a)I]r = -kx$ and $r^t x = 0$ is $r = 0$, as is easily deduced upon taking the inner product of both sides of the first equation with r. Then Proposition 2.2 and (4.25) together imply that $\lambda(\cdot)$ is Lipschitz continuous near any $a \in C(1)$.

Now, for any $r \in M^1(\lambda(a), x)$, one has

$$1 = r^t x = -kx^t \left[B - \lambda(a)I\right]^{-1} x.$$

Then

$$[B - \lambda(a)I]\, r = -kx = \frac{1}{x^t \left[B - \lambda(a)I\right]^{-1} x} x.$$

Equivalently,

$$r = \frac{1}{x^t \left[B - \lambda(a)I\right]^{-1} x} [B - \lambda(a)I]^{-1} x.$$

Upon using $x = \left[B - \lambda(a)I\right]^{-1} a$, we obtain the singleton

$$\begin{aligned} M^1(\Sigma^a) &= \frac{1}{a^t \left[B - \lambda(a)I\right]^{-3} a} \left[B - \lambda(a)I\right]^{-2} a \\ &= \partial(-\lambda)(a), \end{aligned}$$

the latter equality being due to (4.26). The asserted continuous differentiability now follows from Proposition 2.1 (2)(c), with the equation (4.28) being immediate. Then (4.29) follows from the fact that $\lambda'(a; v) = \langle \nabla\lambda(a), v\rangle$. ∎

REMARK 4.1. It is interesting to note that (4.28) can be heuristically derived by differentiating the equation

$$1 - a^t [B - \lambda(a)I]^{-2} a = 0$$

with respect to a, via the chain rule .

We now can show that the generalized gradient of $\lambda(\cdot)$ is nonempty at all points.

COROLLARY 4.2.

$$\partial\lambda(a) \neq \phi \qquad \forall a \in R^n.$$

Proof: Corollary 4.1 takes care of the assertion for those points $a \in C(1)$. Now, if $a \in C(2)$, we have $\lambda(a) = b_1$, which implies that a minimizes $\lambda(\cdot)$ over R^n. This yields a nonempty generalized gradient at such vectors a, by part (1) of Proposition 2.1. ∎

It is useful to have available the simplified formula for $M^l(\lambda, x)$ one obtains when $a = 0$. This is provided in the following lemma. [Along with the range notation $R(\cdot)$, we denote the nullspace by $N(\cdot)$.]

LEMMA 4.1. *Let $a = 0$. Then (λ, x) is feasible for $\tilde{P}(0)$ if and only if λ is an eigenvalue of B and x is a unit length eigenvector belonging to λ. In this case one has*

$$M^l(\lambda, x) = \{r \in N(B - \lambda I) : r^t x = l\}. \tag{4.30}$$

Proof: The claim concerning feasibility is immediate. From the definition of the multiplier set $M^l(\lambda, x)$, we see that $kx \in R(B - \lambda I)$. However, $(B - \lambda I)x = a = 0$ implies $kx \in N(B - \lambda I)$. Since

$$R(B - \lambda I) \cap N(B - \lambda I) = \{0\},$$

the formula (4.30) follows. ■

Following Clarke [3, 2], we shall say that a locally Lipschitz function $f : R^n \to R$ is *regular* at x provided that the (ordinary) directional derivative $f'(x; v)$ exists for every direction v, and $f'(x; v) = f^o(x; v)$.

When b_1 is a simple eigenvalue of B, then Lipschitz continuity of $\lambda(\cdot)$ near 0 and regularity at 0 are assured. This is taken up in the next corollary, which also provides a containment involving $\partial\lambda(0)$ and a formula for the directional derivative. These descriptions are in terms of the set

$$X(0) = \{x \in N(B - b_1 I) : x^t x = 1\}$$

that is, the set of unit length eigenvectors of B belonging to the eigenvalue b_1. Note that when b_1 is simple, the eigenspace $N(B - b_1 I)$ has dimension equal to 1, and $X(0)$ consists of only two vectors. (The proof of part (2) closely parallels that of Corollary 4 of Theorem 6.5.2 in Clarke [2].)

COROLLARY 4.3. *Assume that the b_1 is a simple eigenvalue of B. Then we have the following:*

1. *The function $\lambda(\cdot)$ is Lipschitz of rank K near 0 for any $K > 1$, and*

$$\partial\lambda(0) \subset \text{co}\{X(0)\}. \tag{4.31}$$

2. *Furthermore, $\lambda(\cdot)$ is regular at 0, and for each direction $v \in R^n$ we have*

$$\begin{aligned} \lambda'(0; v) &= \max\{v^t x : x \in X(0)\} \\ &= \max\{v^t x : x \in \text{co}\{X(0)\}\}. \end{aligned}$$

Proof of Part 1: It is readily noted that

$$\Sigma^0 = \{(b_1, x) : x \in X(0)\}.$$

Since the dimension of the eigenspace $N(B - b_1 I)$ is 1, Lemma 4.1 implies

$$M^0(\Sigma^0) = \{0\} \tag{4.32}$$

and

$$M^1(\Sigma^0) = X(0). \tag{4.33}$$

Then (4.25) and Proposition 2.2 imply that $-\lambda(\cdot)$ and therefore also $\lambda(\cdot)$ are Lipschitz near 0, and that

$$\partial(-\lambda)(0) \subset \text{co}\{X(0)\}.$$

Now (4.31) follows from the fact that $\partial(-\lambda)(0) = -\partial\lambda(0)$ [by Proposition 2.1 (2)(e)] and the symmetry about the origin of $\text{co}\{X(0)\}$.

Since 1 is a norm bound on $\partial\lambda(0)$, it follows from Proposition 2.1 2(b) that

$$|\lambda^o(0; v)| \leq ||v|| \qquad \forall v \in R^n.$$

This implies

$$\sup_{\{v:||v||=1\}} \left\{ \left| \limsup_{\substack{a \to 0 \\ t \downarrow 0}} \frac{\lambda(a + tv) - \lambda(a)}{t} \right| \right\} \leq 1. \tag{4.34}$$

We claim that

$$\left| \limsup_{\substack{a^1 \to 0 \\ a^2 \to 0}} \frac{\lambda(a^1) - \lambda(a^2)}{||a^1 - a^2||} \right| \leq 1. \tag{4.35}$$

Suppose to the contrary that (4.35) did not hold. Then there would exist $\delta > 0$ and sequences $t_i \downarrow 0$, $a_i \to 0$, and $v_i \to v$ (with $||v_i|| = 1$) such that

$$\limsup_{t \downarrow 0} \frac{\lambda(a_i + t_i v_i) - \lambda(a_i)}{t_i} \geq 1 + \delta. \tag{4.36}$$

The continuity of $\lambda(\cdot)$ then implies that there exists a unit length vector v such that

$$\limsup_{t \downarrow 0} \frac{\lambda(a_i + t_i v) - \lambda(a_i)}{t_i} \geq 1 + \delta, \tag{4.37}$$

which contradicts (4.34); hence (4.35) holds. Since

$$\left|\limsup_{\substack{a^1\to 0\\ a^2\to 0}} \frac{\lambda(a^1)-\lambda(a^2)}{||a^1-a^2||}\right| = \limsup_{\substack{a^1\to 0\\ a^2\to 0}} \frac{|\lambda(a^1)-\lambda(a^2)|}{||a^1-a^2||}, \tag{4.38}$$

any $K > 1$ can serve as the Lipschitz rank of $\lambda(\cdot)$ near 0.

Proof of Part 2: Fix $x \in X(0)$, and define

$$\lambda_*(a) := \max\{\lambda - ||y-x||^2 : (B-\lambda I)y = a\ , y^t y = 1\}.$$

It is clear that the set of points where the above maximum is attained is

$$\Sigma_*^0 = \{b_1, x\}.$$

We now introduce new multiplier sets for $l \in R$:

$$M_*^l(\lambda, y) = \{r \in R^n : \exists k \in R \text{ s.t. } (B-\lambda I)r + ky + 2(y-x) = 0,\ r^t y = l\}.$$

If one makes the obvious modifications, the proof of Theorem 4.1 goes through with $\lambda_*(\cdot)$ replacing $\lambda(\cdot)$. Analogs of (4.25) and (4.26) result. Note that now

$$M_*^1(\Sigma_*^0) = \{x\} \tag{4.39}$$

and

$$M_*^0(\Sigma_*^0) = \{0\}. \tag{4.40}$$

From the aforementioned analog of Theorem 4.1, Proposition 2.1 (2)(b), and (4.39), we obtain (for any direction $v \in R^n$)

$$\limsup_{t\downarrow 0} \frac{-\lambda_*(tv)+b_1}{t} \le (-\lambda_*)^o(0;v) \le v^t x.$$

Now, using the fact that $\lambda_*(\cdot) \le \lambda(\cdot)$, we get

$$\limsup_{t\downarrow 0} \frac{-\lambda(tv)+b_1}{t} \le \limsup_{t\downarrow 0} \frac{-\lambda_*(tv)+b_1}{t},$$

and therefore

$$\limsup_{t\downarrow 0} \frac{-\lambda(tv)+b_1}{t} \le v^t x.$$

In view of the fact that $x \in X(0)$ was arbitrary, we arrive at

$$\limsup_{t\downarrow 0} \frac{-\lambda(tv)+b_1}{t} \le \min\{v^t x : x \in X(0)\}. \tag{4.41}$$

Because $X(0)$ is symmetric about 0, we can rewrite (4.41) as

$$\liminf_{t\downarrow 0} \frac{\lambda(tv) - b_1}{t} \geq \max\{v^t x : x \in X(0)\}. \tag{4.42}$$

The maximum of a linear function over a compact convex set with nonempty interior is attained on the boundary of the set, since the boundary contains the extreme points. Hence (4.42) leads to

$$\liminf_{t\downarrow 0} \frac{\lambda(tv) - b_1}{t} \geq \max\{v^t x : x \in \operatorname{co}\{X(0)\}\}. \tag{4.43}$$

Now, in view of Proposition 2.1 (2)(b) and the definition of the generalized directional derivative, we have

$$\limsup_{t\downarrow 0} \frac{\lambda(tv) - b_1}{t} \leq \lambda^o(0; v) \leq \max\{v^t x : x \in \operatorname{co}\{X(0)\}\}. \tag{4.44}$$

Together, (4.43) and (4.44) complete the proof. ∎

The following result improves upon part (1) of Corollary 4.3 in two ways. Firstly, the largest eigenvalue of B is permitted to have arbitrary multiplicity. Secondly, the containment (4.31) is sharpened to an equality.

THEOREM 4.2. *The function $\lambda(\cdot)$ is Lipschitz of rank K near 0 for any $K > 1$, and*

$$\partial\lambda(0) = \operatorname{co}\{X(0)\}. \tag{4.45}$$

The proof of the theorem will rely upon several lemmas. In the first of these, we show that it may without loss of generality be assumed that B is diagonal.

LEMMA 4.2. *In proving Theorem 4.2, it is sufficient to consider only the case where*

$$B = \operatorname{diag}\{b_1, b_2, \ldots, b_n\}.$$

Proof: We first consider the claim regarding Lipschitz continuity. If K is a Lipschitz constant for $\hat{\lambda}(\cdot)$ near 0, then for any vectors a^1 and a^2 of sufficiently small norm, (3.19) implies

$$\begin{aligned} |\lambda(a^1) - \lambda(a^2)| &= |\hat{\lambda}(U^t a^1) - \hat{\lambda}(U^t a^2)| \\ &\leq K\|U^t a^1 - U^t a^2\| \\ &= K\|a^1 - a^2\|. \end{aligned}$$

To prove the rest of the lemma, let $a \in C(1)$. Then by the chain rule we have

$$\nabla\lambda(a) = U \nabla \hat{\lambda}(U^t a).$$

Upon letting $a \to 0$, Proposition 2.1 (2)(d) yields

$$\partial\lambda(0) = U\partial\hat{\lambda}(0).$$

Now, if (4.45) is valid for $\hat{\lambda}(\cdot)$, it follows that

$$\begin{aligned}\partial\lambda(0) &= U\{q \in N(\hat{B} - b_1 I) : q^t q \leq 1\} \\ &= U\{q \in N(U^t BU - b_1 I) : q^t q \leq 1\} \\ &= \{r \in N(B - b_1 I) : r^t r \leq 1\}.\end{aligned}$$

■

So let us now assume that B is diagonal, and for $\varepsilon > 0$, define the matrix

$$B_\varepsilon := \mathrm{diag}\{b_1 + \varepsilon, b_2, \ldots, b_n\},$$

which results from perturbing the first diagonal entry of B by ε. (The introduction of B_ε will prove to be useful, because it will enable us to employ Corollary 4.3.) For a given vector $a \in R^n$, we denote the maximal extended eigenvalue of B_ε by

$$\lambda_\varepsilon(a) := \max\{\lambda : \exists x \in R^n \text{ s.t. } (B_\varepsilon - \lambda I)x = a,\ x^t x = 1\}.$$

Let us denote by $P_\varepsilon(a)$ the variant of the mathematical programming problem $P(a)$ one obtains when B is replaced by B_ε, that is,

$$\begin{array}{ll}\text{maximize} & x^t B_\varepsilon x - 2a^t x \\ \text{subject to} & x^t x = 1.\end{array}$$

We shall require the following lemma.

LEMMA 4.3. *Let $a \in R^n$. Then for $\varepsilon > 0$ chosen sufficiently small, one has*

$$|\lambda(a) - \lambda_\varepsilon(a)| \leq \varepsilon \qquad \forall a \in R^n. \tag{4.46}$$

Proof: We will consider the cases $a \in C(1)$ and $a \in C(2)$ separately.

$a \in C(1)$: In this case, $\lambda(a) > b_1$. If $a_1 \neq 0$, then the corresponding secular function for $P_\varepsilon(a)$ is

$$s_a^\varepsilon(\lambda) := 1 - \frac{a_1^2}{(b_1 + \varepsilon - \lambda)^2} - \sum_{j=2}^{n} \frac{a_j^2}{(b_j - \lambda)^2}. \tag{4.47}$$

If $a_1 = 0$, then

$$s_a^\varepsilon(\lambda) := 1 - \sum_{j=2}^{n} \frac{a_j^2}{(b_j - \lambda)^2}. \tag{4.48}$$

Let Γ denote a compact interval which contains $\lambda(a)$ in its interior. We take Γ sufficiently tight around $\lambda(a)$ so as to ensure that none of the b_j is in Γ. We have

$$s_a^\varepsilon(\lambda) \to s_a(\lambda) \quad \text{as} \quad \varepsilon \to 0$$

for each $\lambda \in \Gamma$. Consequently, $s_a^\varepsilon(\cdot)$ approximates $s_a(\cdot)$ uniformly on Γ, to an arbitrary tolerance, depending on how small we choose ε. Now, $s_a(\lambda(a)) = 0$, $s_a'(\lambda(a)) > 0$, and both functions $s_a^\varepsilon(\cdot)$ and $s_a(\cdot)$ are analytic on Γ. Hence, if $\varepsilon > 0$ is chosen sufficiently small, then $s_a^\varepsilon(\hat{\lambda}_\varepsilon) = 0$ for some $\hat{\lambda}_\varepsilon$ as close to $\lambda(a)$ as we specify. In particular, $\hat{\lambda}_\varepsilon > b_1 + \varepsilon$; that is, case 1 holds for $P_\varepsilon(a)$. This implies that $\hat{\lambda}_\varepsilon = \lambda_\varepsilon(a)$. We can now show that $\lambda_\varepsilon(a) \geq \lambda(a)$. Indeed, if $\lambda_\varepsilon(a) < \lambda(a)$, then we get a contradiction because the equalities $s_a(\lambda(a)) = 0$ and $s_a^\varepsilon(\lambda_\varepsilon(a)) = 0$ could not simultaneously hold. In fact, for small $\varepsilon > 0$, we are assured that

$$\lambda_\varepsilon(a) \geq \lambda(a) > b_1 + \varepsilon > b_1. \tag{4.49}$$

We now claim that

$$\lambda_\varepsilon(a) \leq \lambda(a) + \varepsilon. \tag{4.50}$$

In order to verify (4.50), first consider the possibility that $a_1 = 0$. Then the functions $s_a(\cdot)$ and $s_a^\varepsilon(\cdot)$ are identical, and $\lambda(a) = \lambda_\varepsilon(a)$. Hence (4.50) holds if $a_1 = 0$. Now assume that $a_1 \neq 0$, and note that (4.49) implies

$$\frac{a_j^2}{[b_j - \lambda(a)]^2} \geq \frac{a_j^2}{[b_j - \lambda_\varepsilon(a)]^2} \qquad \forall j = 2, 3, \ldots, n. \tag{4.51}$$

Then in order for the equalities $s_a(\lambda(a)) = 0$ and $s_a^\varepsilon(\lambda_\varepsilon(a)) = 0$ to both hold, we must have

$$\frac{a_1^2}{[b_1 - \lambda(a)]^2} \leq \frac{a_1^2}{[b_1 + \varepsilon - \lambda_\varepsilon(a)]^2}. \tag{4.52}$$

Since $a_1 \neq 0$, (4.52) implies that (4.50) holds, from which (4.46) immediately follows.

$a \in C(2)$: In this case, there does not exist $\lambda > b_1$ such that $s_a(\lambda) = 0$. This implies $\lambda(a) = b_1$, and $J(a) \cap J(b) = \phi$. Since $a_1 = 0$, it follows that $s_a(\cdot)$ and $s_a^\varepsilon(\cdot)$ are identical. Consequently, there does not exist $\lambda > b_1 + \varepsilon$ such that $s_a^\varepsilon(\lambda) = 0$. This means that case 2 holds in the problem $P_\varepsilon(a)$.

Hence $\lambda_\varepsilon(a) = b_1 + \varepsilon$, and again (4.46) holds. This concludes the lemma's proof. ∎

The next lemma is an immediate consequence of Corollary 4.3. (Note that it does not require smallness of ε.)

LEMMA 4.4. *Let $\varepsilon > 0$ be given. Then the function $\lambda_\varepsilon(\cdot)$ is Lipschitz of rank K near* 0, *for any $K > 1$.*

We can now complete the proof of Theorem 4.2.

Proof of Theorem 4.2: For any vectors a^1 and a^2 and any $\varepsilon > 0$, the triangle inequality yields

$$\begin{aligned}|\lambda(a^1) - \lambda(a^2)| \;\leq\; & |\lambda_\varepsilon(a^1) - \lambda_\varepsilon(a^2)| \\ & + |\lambda(a^1) - \lambda_\varepsilon(a^1)| + |\lambda(a^2) - \lambda_\varepsilon(a^2)|.\end{aligned}$$

Let $K > 1$. Then Lemmas 4.3 and 4.4 imply

$$|\lambda(a^1) - \lambda(a^2)| \leq K|a^1 - a^2| + 2\varepsilon \tag{4.53}$$

for any a^1 and a^2 of sufficiently small norm, independently of ε. Since ε is arbitrary, this establishes the Lipschitz continuity of rank K for $\lambda(\cdot)$ near 0.

We now turn to the verification of (4.45). Suppose that $0 \neq a \in N(B - b_1)I$. Then the last $n - k$ components of a are 0, where k is the algebraic multiplicity of b_1. Since $a \in C(1)$, we may apply Corollary 4.1. By direct substitution,

$$\nabla \lambda(a) = -\frac{1}{||a||^2}\,[B - \lambda(a)I]\,a. \tag{4.54}$$

Upon using (3.14), we obtain

$$\nabla \lambda(a) = \frac{a}{||a||} \in X(0). \tag{4.55}$$

Letting $a \to 0$, Proposition 2.1 (2)(d) implies

$$\mathrm{co}\{X(0)\} \subset \partial\lambda(0). \tag{4.56}$$

The reverse containment follows from (4.26) (with $a = 0$), the fact that $\partial(-\lambda)(0) = -\partial\lambda(0)$, and

$$M^1(\Sigma^0) = X(0), \tag{4.57}$$

which is a consequence of Lemma 4.1. Hence (4.45) holds, and the proof is complete. ∎

In view of Proposition 3.1, the following corollary is immediate.

COROLLARY 4.4.

$$\partial V(0) = -2\partial\lambda(0). \tag{4.58}$$

We are now in position to establish regularity of $\lambda(\cdot)$ at the origin, without any assumptions on the multiplicity of b_1.

THEOREM 4.3. *The function $\lambda(\cdot)$ is regular at 0. Furthermore, for any vector $v \in R^n$, the directional derivative of $\lambda(\cdot)$ at 0 in the direction v is given by*

$$\begin{aligned}\lambda'(0;v) &= \max\{v^t x : x \in X(0)\}\\ &= \max\{v^t x : x \in \mathrm{co}\{X(0)\}\}.\end{aligned}$$

Proof: We first point out that no generality is lost if we assume that B is diagonal. One uses arguments very similar to the proof of Lemma 4.2; we leave the details to the reader.

We know that $\lambda_\varepsilon(a) \le \lambda(a) + \varepsilon$ and $\lambda_\varepsilon(0) = b_1 + \varepsilon$. We also know that $\lambda_\varepsilon(a)$ is regular at 0, and $\lambda'_\varepsilon(0;v) = |r^t v|$, where r is either of the two unit length eigenvectors of B_ε belonging to the eigenvalue $b_1+\varepsilon$; note that these are also eigenvectors of B belonging to b_1. Therefore,

$$\begin{aligned}\liminf_{t\downarrow 0}\frac{\lambda(tv)-\lambda(0)}{t} &\ge \liminf_{t\downarrow 0}\frac{\lambda_\varepsilon(tv)-\lambda_\varepsilon(0)}{t}\\ &= |r^t v|.\end{aligned}$$

Since the above relations are true if we define B_ε via an ε-perturbation in *any* diagonal entry of B equalling b_1, we see that

$$\begin{aligned}\liminf_{t\downarrow 0}\frac{\lambda(tv)-\lambda(0)}{t} &\ge \max\{v^t x : x \in X(0)\}\\ &= \max\{v^t x : x \in \mathrm{co}\{X(0)\}\} \ge \lambda^o(0;v).\end{aligned}$$

On the other hand, by definition we have

$$\limsup_{t\downarrow 0}\frac{\lambda(tv)-\lambda(0)}{t} \le \lambda^o(0;v). \tag{4.59}$$

This completes the proof. ∎

5. CONCLUDING COMMENTS

Most of our results have focused on the behavior of $\lambda(\cdot)$ near the origin. It would be of interest to ascertain the local behavior of this function also at nonzero points. Points in $C(1)$ present no mystery, since we have continuous differentiability on that set, and a formula for the gradient. However, describing the behavior of $\lambda(\cdot)$ near a general nonzero point in $C(2)$ remains an open problem. Other questions which remain to be answered concern establishing "global" properties of $\lambda(\cdot)$. (Recall that in Remark 3.3 we noted that neither convexity nor concavity holds, in general.) One global property that can be verified is a certain kind of monotonicity. Before giving the result, we need to introduce some further notation: Given a vector $x \in R^n$, we denote by $|x|$ the vector whose i^{th} component is $|x_i|$, $i = 1, 2, \ldots, n$. The inequality $|x| \geq |y|$ means that $|x_i| \geq |y_i|$ holds for each component.

PROPOSITION 5.1. *Let U be a unitary matrix such that*

$$U^t B U = \operatorname{diag}\{b_1, b_2, \ldots, b_n\}.$$

Then the following hold:

$$|Ua| = |U\bar{a}| \quad \Longrightarrow \quad \lambda(Ua) = \lambda(U\bar{a}). \tag{5.60}$$

$$|Ua| > |U\bar{a}| \quad \Longrightarrow \quad \lambda(Ua) > \lambda(U\bar{a}). \tag{5.61}$$

Proof: In view of (3.19), it suffices to prove the assertion for B already diagonal; that is, $U = I$. The implication (5.60) is a direct consequence of the definition of $\lambda(\cdot)$. We now turn towards proving (5.61). If $\bar{a} \in C(2)$, then there is nothing to show, since the global minimum value of $\lambda(\cdot)$ is attained only on $C(2)$, and $|a| > |\bar{a}|$ implies that $a \notin C(2)$. So we assume that $\bar{a} \in C(1)$, and that $|a| > |\bar{a}|$. Then $a \in C(1)$, because no component of a is zero. Now, we necessarily have

$$1 - \sum_{j=1}^{n} \frac{a_j^2}{[b_j - \lambda(a)]^2} = 0 \tag{5.62}$$

and

$$1 - \sum_{j=1}^{n} \frac{\bar{a}_j^2}{[b_j - \lambda(\bar{a})]^2} = 0. \tag{5.63}$$

Since $a_j^2 > \bar{a}_j^2$ for each j, the previous two equations imply that $\lambda(a) > \lambda(\bar{a})$. ∎

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Received 22 April 1992; final manuscript accepted 18 September 1992